Application of Interpolation in Networks

H. Jerome Keisler and Jeffrey M. Keisler
Department of Mathematics, University of Wisconsin-Madison,
Department of Management Science and Information Systems,
University of Massachusetts-Boston

1. Origin of Our Work

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The motivation: Imagine an organization with agents who each have a technical vocabulary and knowledge base, and with links that can carry messages from one agent to another in their common vocabulary. After receiving messages from other agents, one of the agents has the task of deciding between a set of alternatives.

Our aim is to understand when such an organization will be able to arrive at the "right" decision— the decision that would be made by a hypothetical oracle who has access to the knowledge base of every agent.

The Craig Interpolation Theorem plays a key role in this work. The large expressive power of first order logic provides a great deal of flexibility in applications. This motivates a rich variety of purely mathematical problems in the area of model theory.

We give a survey of two of our joint papers published in 2012 and 2014, and a future joint paper in preparation.

▶

1. Our first paper concerns a strong provability notion called **report provability**. Consider a finite directed graph with a first order signature and knowledge base at each vertex. When (x, y) is an edge, a sentence C can be reported from x to y provided that that C is in the common language of x and y and is a consequence of the knowledge base of x and the sentences previously reported to x. A report proof of a sentence G at a vertex d is a finite sequence of reports such that G is a consequence of the knowledge base of d and the sentences reported to d. Many distributed computing algorithms in the literature (for example, the junction tree algorithm) can be viewed as report proofs.

2. In our second paper, vertices observe additional sentences before making reports, and the goal is decide between a set of alternatives after the reports. This leads to a family of similar report proofs that depend on the observations.

3. We will conclude with research in progress where the potential observations are well behaved. This work uses the model theory of o-minimal structures.

▶

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By a (simple) **directed graph** we mean a pair (V, E) where V is finite and $E \subseteq \{(x, y) \colon x, y \in V, x \neq y\}$ (i.e., no loops or multiple edges). A vertex $d \in V$ is a **decider** if for each $x \in V$ there is a path from x to d. A **pointed graph** is a directed graph with at least one decider.

Definition

Signature network $\mathbb{L}$: A pointed graph (V, E) equipped with a signature (set of symbols) $L(x)$ at each vertex $x \in V$.

Language of x: The set $[L(x)]$ of first order sentences with signature $L(x)$.

Definition

Knowledge base $\mathbb{K}$ over $\mathbb{L}$: A theory $K(x) \subseteq [L(x)]$ for each $x \in V$. The **combined knowledge base** is the set $K(V) = \bigcup_x K(x)$.

Convention: $\mathbb{L}$ always denotes a signature network, $\mathbb{K}$ a knowledge base over $\mathbb{L}$, and d a decider in $\mathbb{L}$. We often fix $\mathbb{L}$ and d, and let $\mathbb{K}$ vary.

▶

A **report** from x to y in $\mathbb{L}$ is a triple (x, y, C) such that $(x, y) \in E$ and $C \in [L(x) \cap L(y)]$. Verbally, (x, y, C) means "x reports C to y".

Definition

Let $G \in [L(d)]$ (the goal sentence). A **report proof** of G at d in $\mathbb{K}$ is a finite sequence of **reports** $(x_1, y_1, C_1) \ldots (x_k, y_k, C_k)$ in $\mathbb{L}$ such that:

- for each $i \leq k$, C_i is provable from $K(x_i)$ and sentences previously reported to x_i.
- $d = y_k$ and G is provable from $K(d)$ and sentences reported to d.

G is **report provable** (at d in $\mathbb{K}$) if there exists a report proof of G..

Theorem

(Soundness) If G is report provable at d in $\mathbb{K}$, then G is provable from the combined knowledge base $K(V)$.

But sometimes G is provable from $K(V)$ but not report provable at d in $\mathbb{K}$.

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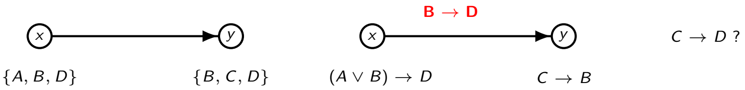
5. The Simplest Case: Two Vertices

The picture shows:
a signature network and knowledge base with just two vertices and
a report proof with just one report.



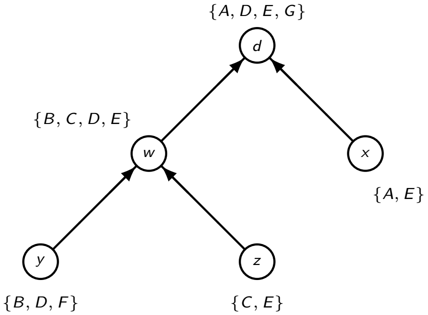
In the present terminology, the Craig Interpolation Theorem says:

Theorem

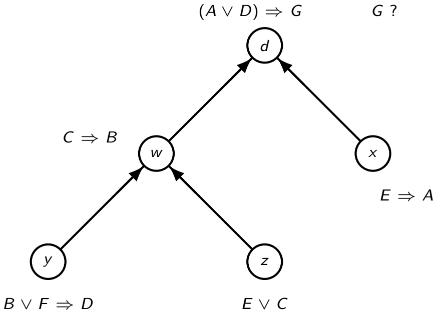
If $\mathbb{L}$ is a signature network with $V = \{x, y\}$ and $E = \{(x, y)\}$, $\mathbb{K}$ is a knowledge base over $\mathbb{L}$, and $G \in [L(y)]$ is provable from the combined knowledge base, then G is report provable at y.

▶

There is a unique decider d, shown at the top.
We will give a report proof of G at d.

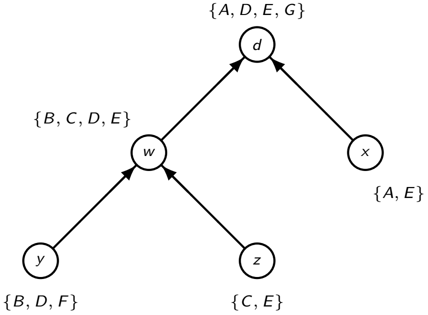


$\mathbb{L}$ is a signature network



Knowledge base $\mathbb{K}$ over $\mathbb{L}$

The reports in the proof are shown in red.



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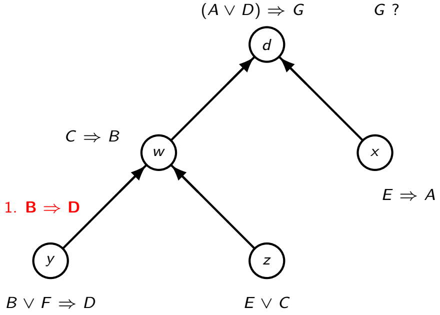

Report proof, step 1

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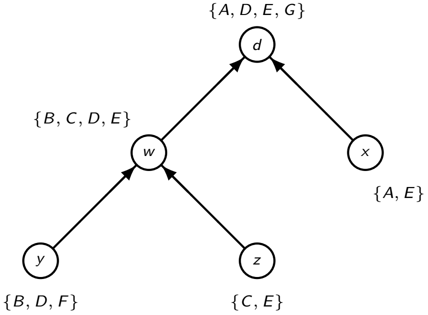

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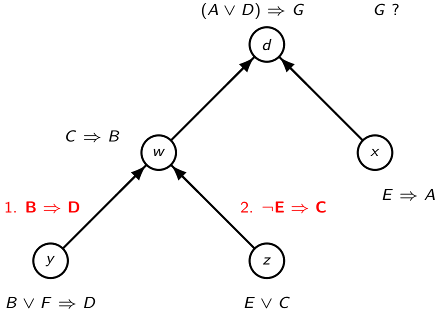

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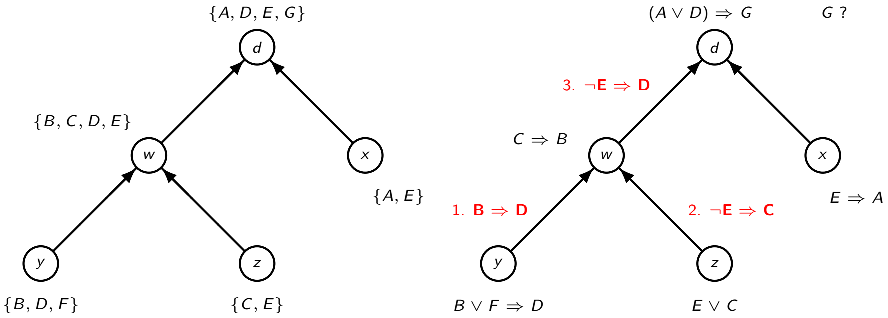
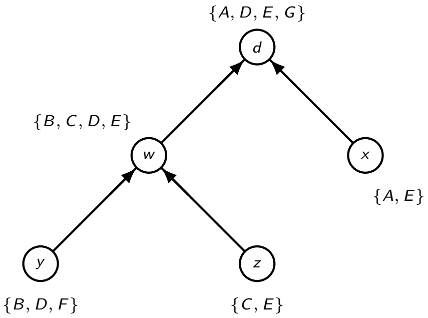


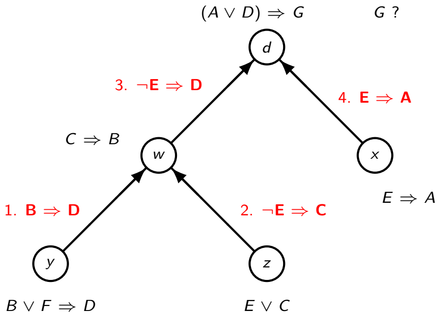
$\mathbb{L}$ is a signature network

Report proof, step 3

The reports in the proof are shown in red.

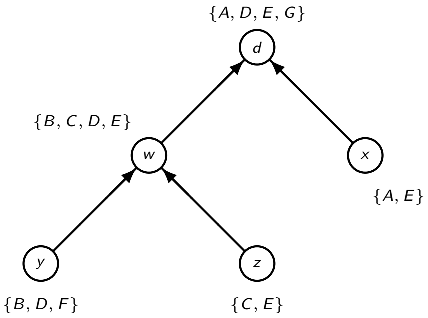


$\mathbb{L}$ is a signature network

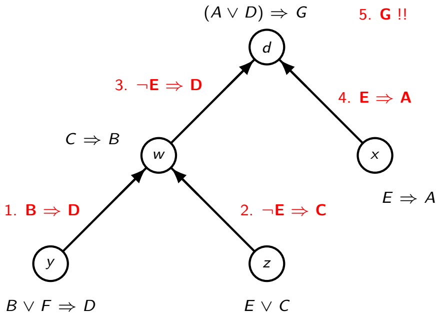


Report proof, step 4

The reports in the proof are shown in red.

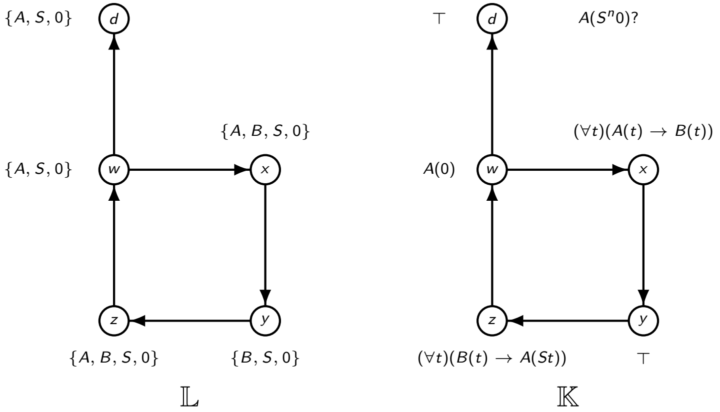


$\mathbb{L}$ is a signature network



Completed report proof

A long report proof. $A(S^n 0)$ is report provable with $4n+1$ reports.
$(\forall t)(A(t) \to A(St))$ is provable from $K(V)$ but not report provable at d.



Definition

$\mathbb{L}$ is **report complete** if for every decider d and every $\mathbb{K}$ over $\mathbb{L}$, every sentence in $[L(d)]$ that is provable from $K(V)$ is report provable at d in $\mathbb{K}$.

Report completeness is a property of the signature network $\mathbb{L}$ alone. It is a guarantee that no matter what the knowledge base is, every consequence of the combined knowledge base has a report proof.

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The Main Question

Which signature networks are report complete?

By the Craig Interpolation Theorem, every signature network with only two vertices is report complete.

-

Definition

$V[\sigma] = (V, E)$ restricted to $\{x \in V : L(x) \supseteq \sigma\}$.
$\mathbb{L}$ has the **Peak Property** if every nonempty $V[\sigma]$ has at least one decider.

We think of $V[\sigma]$ as a mountain peak with its deciders at the summit.

Theorem

If $\mathbb{L}$ is report complete then $\mathbb{L}$ has the Peak Property.

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If $\mathbb{L}$ is report complete then $\mathbb{L}$ has the Peak Property.

Definition

(V, E) is a **tree** if $(\exists d)(\forall x)$ there is a unique path from x to d.
$\mathbb{L}$ is a **signature tree** if $\mathbb{L}$ has the Peak Property and (V, E) is a tree.
$\mathbb{L}$ **contains a signature tree** if $\mathbb{L}$ can be built from a signature tree by adding edges and making no other changes.

▶

10. Properties of Signature Networks

The following properties of signature networks are important:

- RC Report completeness.
- CT Containing a signature tree.
- PP The Peak Property.
- TW The Twin Peaks Property (to be defined later).
- LC The Linked Chain Property (to be defined later).

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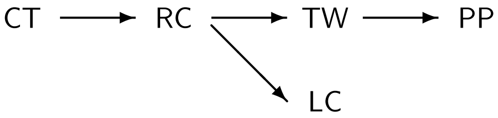
Implications between these properties:



There are counterexamples showing these implications cannot be reversed. Each of these properties is preserved under adding some edges to the graph while leaving the vertices and signatures unchanged.

▶

 11 A Sufficient Condition

Theorem

(Follows from Amir and McIlraith 2006) If $\mathbb{L}$ contains a signature tree, then $\mathbb{L}$ is report complete. That is, CT $\Rightarrow$ RC.

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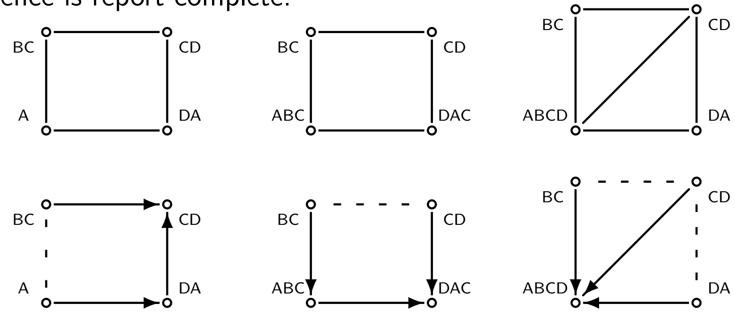
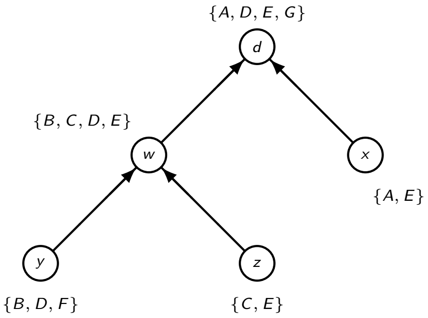
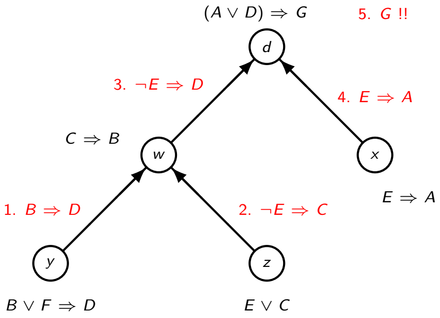
Each signature network below contains a signature tree as shown, and hence is report complete.



$\mathbb{L}$ is a signature tree and hence is report complete.



$\mathbb{L}$ is a signature network

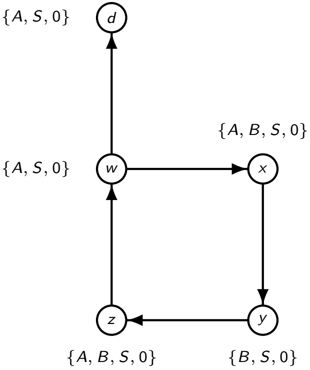


Report proof in a $\mathbb{K}$ over $\mathbb{L}$

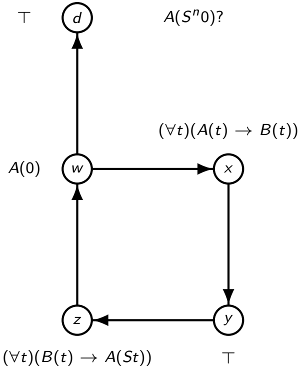
13. Example 2 Again

Peak property fails (because $V[\{A,B\}] = \{z,x\}$ is not connected).
So $\mathbb{L}$ is not report complete.

$(\forall t)(A(t) \to A(St))$ is provable from $K(V)$ but not report provable at d.



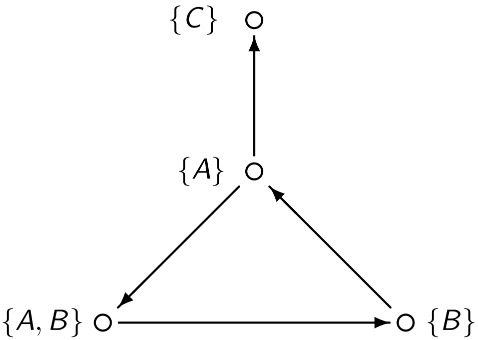
$\mathbb{L}$



$\mathbb{K}$

 14. Counterexample

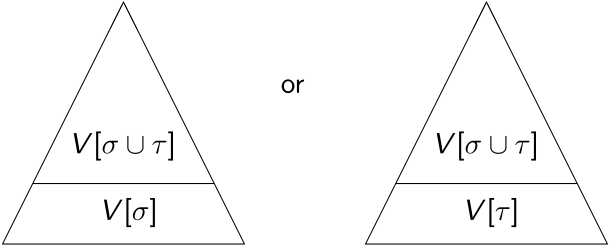
A signature network that is report complete but does not contain a signature tree. So the implication CT $\Rightarrow$ RC is strict.



Definition

$\mathbb{L}$ has the **Twin Peaks Property** if: (1) $\mathbb{L}$ has the Peak Property and (2) For all $\sigma, \tau \subseteq V$, if $V[\sigma \cup \tau] \neq \emptyset$, then either
for each $x \in V[\sigma]$, $V[\sigma]$ contains a path from x to $V[\sigma \cup \tau]$, or
for each $y \in V[\tau]$, $V[\tau]$ contains a path from y to $V[\sigma \cup \tau]$.

(2)



16. Twin Peaks Property Theorems

Theorem

If $\mathbb{L}$ is report complete then $\mathbb{L}$ has the Twin Peaks Property, RC $\Rightarrow$ TW.

Theorem

If $\mathbb{L}$ is on a directed acyclic graph (DAG), then RC $\Leftrightarrow$ TW and RC $\Leftrightarrow$ CT.

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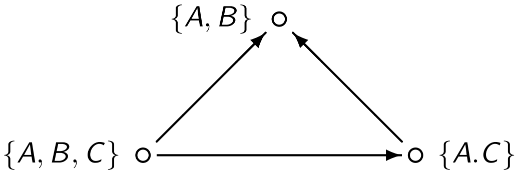
Theorem

If $\mathbb{L}$ is on a directed acyclic graph (DAG), then RC $\Leftrightarrow$ TW and RC $\Leftrightarrow$ CT.

The implication TW $\Rightarrow$ PP is strict, even in the DAG case.
Here is a signature network on a DAG that has the Peak Property but not the Twin Peaks Property, hence is not report complete.



17. Linked Chain Property

Definition

$\mathbb{L}$ has the **Linked Chain Property** if for every $p \geq 2$ and every sequence of disjoint finite sets of symbols $(\sigma_0, \ldots, \sigma_p)$, if each $\sigma_j \cup \sigma_{j+1}$ is contained in some $L(y_j)$ (putting $\sigma_{p+1} = \sigma_0$), then some $L(y)$ contains 3 distinct σ_j's.

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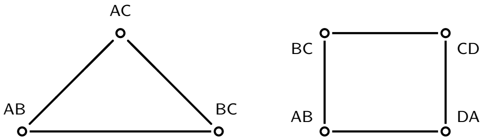
Theorem

If $\mathbb{L}$ is report complete then $\mathbb{L}$ has the Linked Chain Property, $RC \Rightarrow LC$.

The following signature networks do not have the Linked Chain Property, hence are not report complete..



18. Connected (Undirected) Graphs

A **connected graph** (sometimes called a connected undirected graph) is a pointed graph where E is symmetric. Note that every vertex is a decider.

Theorem

If $\mathbb{L}$ is on a connected graph, then $RC \Leftrightarrow LC \wedge PP$ and $RC \Leftrightarrow CT$.

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Corollary

If $\mathbb{L}$ is on a singly connected graph (connected with no cycles of length > 2), then $RC \Leftrightarrow PP$.

The **junction tree algorithm** in the literature is a report proof on a singly connected graph that uses very special first order sentences.

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Corollary

If $\mathbb{L}$ is on a complete graph, the $RC \Leftrightarrow LC$.

▶

19. Discussion

For a signature network $\mathbb{L}$, report completeness guarantees that for *every* knowledge base, each provable sentence is report provable at d.
Our results above give upper bounds on the complexity of report completeness.

To consider complexity, we fix countable sets $\mathcal{V}$ of possible vertices and $\mathcal{L}$ of symbols. Let $\mathcal{N}$ be the countable set of signature networks $\mathbb{L}$ such that V is a finite subset of $\mathcal{V}$ and $L(x)$ is a finite subset of $\mathcal{L}$ for each $x \in V$.

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In general, the set of report complete $\mathbb{L} \in \mathcal{N}$ is Π^0_2. Are there better bounds?

The set of report complete $\mathbb{L} \in \mathcal{N}$ with a DAG is recursive
and even polynomial time computable (PTIME).
Because the set of $\mathbb{L} \in \mathcal{N}$ with the twin peaks property and a DAG is PTIME.

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The set of report complete $\mathbb{L} \in \mathcal{N}$ with a connected graph is EXPTIME.
Because the set of $\mathbb{L} \in \mathcal{N}$ with the linked chain and peak properties is EXPTIME.
We do not know whether that set is PTIME.

▶

We now turn to the situation where the agents first make observations and then report sentences to their neighbors, and some decider d is faced with the problem of deciding between a set of **alternatives** $\mathbb{A} \subseteq [L(d)]$.

Definition

Observation network $\mathbb{O}$: A knowledge base $\mathbb{K}$ over $\mathbb{L}$ equipped with a non-empty set $O(x) \subseteq [L(x)]$ of **potential observations** for each $x \in V$.

Definition

s is a **selection** (actual observation) in $\mathbb{O}$ if $s(x) \in O(x)$ for each $x \in V$.
$\mathbb{K}^s$ is the knowledge base where $K^s(x) = K(x) \cup \{s(x)\}$ for each $x \in V$.
$\mathbb{O}$ is **decisive** for a set $\mathbb{A}$ if $(\forall s)$ some $D \in \mathbb{A}$ is provable from $K^s(V)$.

We consider D to be a correct decision for s if D is provable in $\mathbb{K}^s$.
This gives a whole family of similar decision problems that depend on s.

▶

Corollary

If $\mathbb{L}$ is report complete, $\mathbb{A} \subseteq [L(d)]$, and $\mathbb{O}$ is decisive for $\mathbb{A}$, then $(\forall s)$ some $D \in \mathbb{A}$ is report provable in $\mathbb{K}^s$ at d.

For simplicity, we assume from now on that $\mathbb{O}$ is an observation network such that for each x, $O(x)$ is finite and pairwise inconsistent.
And the underlying graph is a DAG, and thus has a unique decider d.

Our next result shows that if $\mathbb{L}$ is report complete and $\mathbb{O}$ is decisive for $\mathbb{A} \subseteq [L(d)]$, then there is a rule $\mathbb{P}$, called a report plan, such that:

- $\mathbb{P}$ can be prepared in advance before the agents make observations.
- After any selection s, the team of agents can use $\mathbb{P}$ to produce a report proof of some alternative $D \in \mathbb{A}$ at d.
- Each agent x only has to know about its own selection and the reports it receives, not about those of the other agents.

▶

Definition

$\mathbb{P}$ is a **report plan** for $\mathbb{A}$ in $\mathbb{O}$ if for each vertex x:

- For each edge (x, y), $\mathbb{P}(x, y)$ is a finite subset of $[L(x)] \cap [L(y)]$.
- If H meets $O(x)$ once and $\mathbb{P}(z, x)$ once for each edge (z, x), then
  - For each edge (x, y), $K(x) \cup H \vdash C_H$ for some $C_H \in \mathbb{P}(x, y)$.
  - If $x = d$ then $K(x) \cup H \vdash D_H$ for some $D_H \in \mathbb{A}$.
  - $\mathbb{P}$ also specifies C_H and D_H.

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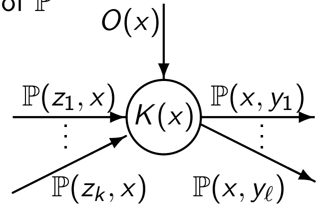
x sees only this part of $\mathbb{P}$



Hereafter we assume that $\mathbb{L}$ is report complete.

Theorem

If $\mathbb{O}$ is decisive for $\mathbb{A} \subseteq [L(d)]$ then there is a report plan for $\mathbb{A}$ in $\mathbb{O}$.

Theorem

If $\mathbb{P}$ is a report plan for $\mathbb{A}$ in $\mathbb{O}$ then for every selection s there is a report proof of some $D \in \mathbb{A}$ in $\mathbb{K}^s$ at d such that for each edge (x, y), every report from x to y belongs to $\mathbb{P}(x, y)$,

23. Report Plans, Existence

Hereafter we assume that $\mathbb{L}$ is report complete.

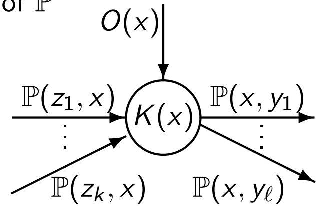
Theorem

If $\mathbb{O}$ is decisive for $\mathbb{A} \subseteq [L(d)]$ then there is a report plan for $\mathbb{A}$ in $\mathbb{O}$.

Theorem

If $\mathbb{P}$ is a report plan for $\mathbb{A}$ in $\mathbb{O}$ then for every selection s there is a report proof of some $D \in \mathbb{A}$ in $\mathbb{K}^s$ at d such that for each edge (x, y), every report from x to y belongs to $\mathbb{P}(x, y)$,

x uses only this part of $\mathbb{P}$



Sometimes it is simpler to focus on a restricted class of sentences.

Suppose $\mathbb{O}$ is an observation network that is decisive for $\mathbb{A}$.

If $O(x)$, $K(x)$, and $\mathbb{A}$ are sets of quantifier-free sentences for each vertex x, then there is a report plan $\mathbb{P}$ for $\mathbb{A}$ such that each $\mathbb{P}(x, y)$ is a set of quantifier-free sentences. Similarly for propositional sentences.

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We often use a new constant symbol c for an unknown quantity.
Suppose $K(d)$ contains the axioms for ordered fields.
Let $\varepsilon > 0$. and $\mathbb{P}$ be a report plan for the set of alternatives

$$\mathbb{A} = \{r \leq c \leq r + \varepsilon \colon r \text{ rational } \}.$$

Then for each selection s, $\mathbb{P}$ produces a report proof that gives the approximate value of c within ε.

Similarly for n-tuples of new constant symbols.

▶

An observation network $\mathbb{O}$ is a knowledge base over $\mathbb{L}$ equipped with a non-empty set $O(x) \subseteq [L(x)]$ of potential observations for each vertex x.

We assume that each $O(x)$ is pairwise inconsistent and $\mathbb{L}$ has a DAG. An actual observation s picks an $s(x) \in O(x)$ for each x.

The decider d is faced with the task of deciding among a set of alternative sentences $\mathbb{A} \subseteq [L(d)]$.

There is a family of similar knowledge bases $\mathbb{K}^s$ obtained by adding $s(x)$ to $K(x)$ for each x.

A report plan for $\mathbb{A}$ in $\mathbb{O}$ is a rule that is local for each x and produces a report proof of some $D \in \mathbb{A}$ in $\mathbb{K}^s$ for each s.

If $\mathbb{O}$ is decisive for $\mathbb{A}$, then there is a report plan for $\mathbb{A}$ in $\mathbb{O}$.

▶

We now survey some work in progress. We discuss cases where the potential observations are simple statements about possible values of unknown quantities.

For a structure M, L_M is the signature of M plus a constant symbol for each element of M. $\mathbb{C} = \{c_0, c_1, \ldots\}$ is a set of new constant symbols.
Each sentence $B \in [L_M \cup \mathbb{C}]$ with n new constants defines an n-ary relation on M. Such a relation is said to be **definable in** M.

Definition

An **o-minimal structure** over $\mathbb{R}$ is a structure $M = (\mathbb{R}, <, \ldots)$ such that each definable $X \subseteq \mathbb{R}$ is a finite union of points and open intervals,

Each of the following is o-minimal over $\mathbb{R}$.

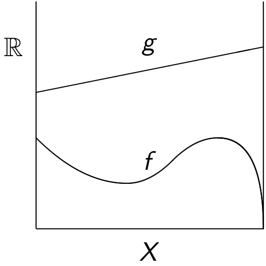
$\mathbb{R}_{lin} = (\mathbb{R}, <, +, s_r)_{r \in \mathbb{R}}$ where $s_r(\cdot)$ is scalar multiplication by r.
$\mathbb{R}_{alg} = (\mathbb{R}, <, 0, 1, +, \times)$ (Tarski 1951).
$\mathbb{R}_{exp} = (\mathbb{R}, <, 0, 1, +, \times, \exp)$ (Wilkie 1996).

$\mathbb{R}_{an} = (\mathbb{R}, <, 0, 1, +, \times, \ldots)$ with a symbol for each analytic function restricted to the unit n-cube (Van den Dries, Macintyre, Marker 1994).

▶

The notion of a **cell** (due to Van den Dries) is an *n*-dimensional analogue of an interval in $\mathbb{R}$. A cell $X \subseteq \mathbb{R}$ is a point or open interval.

For each cell $X \subseteq \mathbb{R}$ and definable continuous $f, g \colon X \to \mathbb{R}$ with $f < g$, the following subsets of $\mathbb{R}^2$ are cells.



$Y_1 = X \times \mathbb{R}$

$Y_2 =$ the graph of f

$Y_3 =$ the region strictly below f

$Y_4 =$ the region strictly above f

$Y_5 =$ the region strictly between f and g

Cells Y in $\mathbb{R}^{n+1}$ are defined in a similar way from cells X in $\mathbb{R}^n$.
Every cell is definable and connected. Each point is a cell.

▶

28. Observations can be o-minimal Cells

Suppose M is o-minimal over $\mathbb{R}$.
We say that $\mathbb{O}$ is **simply decisive** for $\mathbb{A}$ if

- $\mathbb{O}$ has a report plan for $\mathbb{A}$.
- For each x, $L(x) = L_M \cup \mathbb{C}_x$ where $\mathbb{C}_x$ is a finite subset of $\mathbb{C}$.
- The sentences in $O(x)$ define a partition of $\mathbb{R}^n$ where $n = |C_x|$.
- For each x, $K(x)$ contains at least every sentence in $[L_M]$ true in M.

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- The sentences in $O(x)$ define a partition of $\mathbb{R}^n$ where $n = |C_x|$.
- For each x, $K(x)$ contains at least every sentence in $[L_M]$ true in M.

Theorem

If $\mathbb{O}$ is simply decisive for $\mathbb{A}$, there is an observation network $\mathbb{O}^\#$, called a **refinement** *of $\mathbb{O}$, such that $\mathbb{O}^\#$ is also simply decisive for $\mathbb{A}$ and for each x,*

- *Each sentence in $O^\#(x)$ defines a cell.*
- *$\mathbb{O}^\#$ has the same signature network and knowledge base as $\mathbb{O}$*
- *The partition defined by $O^\#(x)$ refines the partition defined by $O(x)$.*
- *If $D \in O^\#(x)$, $B \in O(x)$, $D \vdash B$, every $c \in \mathbb{C}$ occurring in D occurs in B.*

In $\mathbb{R}_{lin}$, the simplest definable functions are the **affine functions** $f(c_1, \ldots, c_n) = r_0 + r_1c_1 + \cdots + r_nc_n$.

Definition

In $\mathbb{R}_{lin}$, a **semilinear cell** is defined in the same way as a cell but using affine functions instead of definable continuous functions.

Each semilinear cell is a cell in $\mathbb{R}_{lin}$.
But there are also much more complicated cells in $\mathbb{R}_{lin}$.
Every definable relation is a finite union of semilinear cells.

Theorem

Suppose $M = \mathbb{R}_{lin}$ and $\mathbb{O}$ is simply decisive for $\mathbb{A}$. Then $\mathbb{O}$ has a refinement $\mathbb{O}^{\#}$ such that for each x, every sentence in $O^{\#}(x)$ defines a semilinear cell, and is a finite conjunction of linear inequalities.

▶

Conclusion

1. Properties of signature networks alone imply things about what can be done when sentences in the common language are passed along edges.

2. Results can be viewed as generalizations of the Craig Interpolation Theorem to networks.

3. Interesting questions arise on the borderline between model theory and graph theory.

4. This work opens up possibilities for applications in settings where tasks are distributed.

Slides will be made available to the meeting, including some extra slides with additional examples.

email address: keisler@math.wisc.edu

▶

Some References

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The hard part is to assume $\mathbb{L}$ is on a connected graph and has the Peak and Linked Chain Properties, and prove $\mathbb{L}$ contains a signature tree.

Lemma: If any symbol in more than one $L(x)$ is in $L(d)$ (d has a big signature), then $\mathbb{L}$ contains a signature tree.

Use induction on $|V|$.

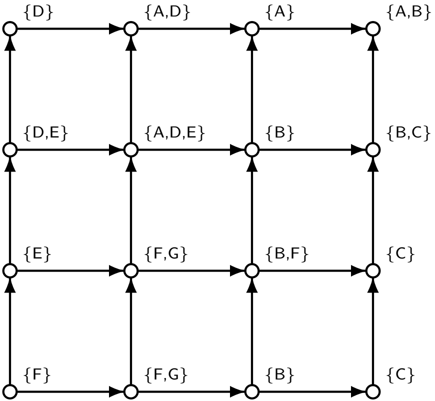
Break $\mathbb{L}$ into finitely many disjoint $\mathbb{L}_i$ such that:
any symbol in more than one $\mathbb{L}_i$ is in $L(d)$,
each $\mathbb{L}_i$ has the Peak and Linked Chain Properties,
and each $\mathbb{L}_i$ has an agent with a big signature.

If there is only one $\mathbb{L}_i$, it contains a signature tree by the lemma.
Otherwise by induction each $\mathbb{L}_i$ contains a signature tree.

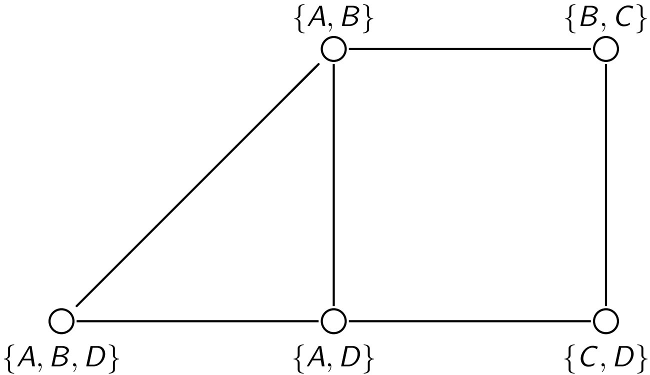
Combine each V_i into a big agent and show that the these big agents form a signature tree. Use these facts to show that $\mathbb{L}$ contains a signature tree.

▶

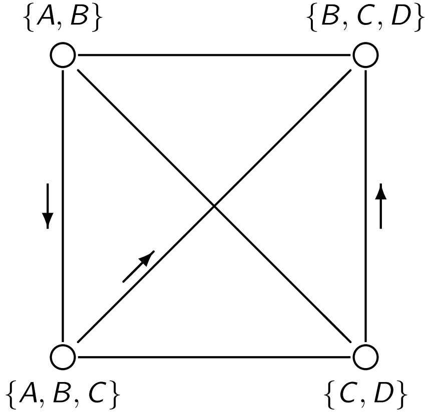
A signature network on a DAG that has the Twin Peaks Property, hence is report complete and contains a signature tree.



A signature network on a connected graph that has the Linked Chain and Peak Properties,
hence is report complete and contains a signature tree.



A signature network on a complete graph that has the Linked Chain Property, hence is report complete and contains a signature tree .



A signature network on a connected graph that has
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hence is report complete and contains a signature tree.